

A UNIFIED APPROACH TO THE THEORY OF CONNECTIONS IN FINSLER GEOMETRY

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ABSTRACT. We propose a unified approach to the theory of connections in the geometry of sprays and Finsler metrics which, in particular, gives a simple explanation of the well-known fact that all the classical Finslerian connections provide exactly the same formulas appearing in the calculus of variations.

Keywords. Finsler metrics, sprays, linear connections, family of affine connections, second variation of energy functional.

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1. INTRODUCTION

Let us start by recalling the following

Definition 1.1. A Finsler metric on a manifold M is a function $F : TM \rightarrow [0, \infty[$, smooth on the slit tangent bundle $TM \setminus 0$, such that

- (1) $F(u) = 0$ if, and only if, $u = 0$;
- (2) $F(\lambda u) = \lambda F(u)$ whenever λ is a positive real number;
- (3) At each $w \in TM \setminus 0$, the fiber-Hessian of F^2 is positive-definite; that is, the *fundamental tensor* $g_w : T_x M \times T_x M \rightarrow \mathbb{R}$ ($w \in T_x M$) of F , defined by

$$g_w(u, v) = \frac{1}{2} \frac{\partial^2}{\partial t \partial s} \Big|_{(0,0)} F(w + su + tv)^2,$$

is a positive-definite inner product on $T_x M$.

The geodesics of a Finsler metric can be defined as the critical points of the energy functionals

$$(1.1) \quad E : \Omega_{p_1, p_2} \longrightarrow \mathbb{R}, \quad E(\lambda) = \frac{1}{2} \int_0^1 F(\dot{\lambda}(t))^2 dt,$$

where Ω_{p_1, p_2} is the space of all regular curves (or piecewise regular) joining p_1 to p_2 . If one aims to compute the second variation of (1.1), or of its two end-manifold analog (3.2), at a critical point, it becomes convenient to dispose of a somehow compatible theory of connections. On the other hand, the extra dependence on directions for the elements involved (caused by the lack of smoothness of F at the null section) implies that the linear connections that arise are naturally defined on (a vector subbundle of) the double tangent bundle of M and there is not a canonical choice of such connection. Instead, there are in the literature many connections, each of them characterized by some kind of compatibility conditions, the most notable ones being due to Berwald, Cartan, Chern and Rund, and Hashiguchi; see, for instance, [2], [3], [4], [7]. We propose here a unified approach to the theory of

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connections which, in particular, makes clearer the well-known fact that all those classical connections provide exactly the same formulas appearing in the calculus of variations. Indeed, we exhibit a mild set of compatibility conditions under which any connection would work just as well.

A linear connection on the double tangent bundle of M may, to a great extent, be thought of as a family of affine connections on M , and this has the advantage of avoiding having to work with tensors on the tangent bundle and, thus, making some computations more similar to the way they are done in Riemannian geometry. Indeed, some authors have approached Finsler geometry through that point of view; see [8], [11], [12] and [9]. The present article works out in full generality the transition from linear connections on the double tangent bundle to families of affine connections, stressing out what conditions a linear connection has to fulfil so as to assure that its corresponding family of affine connections enjoy the nice properties that ultimately lead to the desired computations. For instance, as we show in §2.4.3, a mild nullity condition on the torsion suffices to guarantee that, under suitable assumption, the corresponding affine connections recover the curvature endomorphism of a spray (and, hence, the flag curvature of a Finsler metric); this should be compared with [9], where the author established the same result in the particular case of the Chern-Rund affine connections. Besides, our approach clarifies why some notions such as the covariant derivative, the curvature endomorphism and the second fundamental form appearing in (3.3) do not depend upon the choice of the connection used to define them (see Corollary 2.12, Proposition 2.15 and Proposition 5.3). All these are carried out in §2 and §3, where we have followed a coordinate-free approach in order to make the computations more geometric.

In §4, we discuss briefly the already mentioned Finslerian classical connections. In particular we describe the families of affine connections corresponding to the Cartan and Hashiguchi linear connections, something that apparently was missing from the literature. These descriptions lead to the remarkable conclusion that the Berwald and Hashiguchi linear connections induce the same family of affine connections, and the same is true of the linear connections of Cartan and Chern-Rund. We acknowledge that this fact was first noticed by M. Javaloyes and communicated to the author.

We conclude these notes with an appendix in which we show a symplectic, connection-free, definition of the second fundamental form introduced in §3.3.

2. SPRAYS, CONNECTIONS AND CURVATURE

We begin by considering the theory of linear connections, and their corresponding affine connections, with regard to a given spray on the manifold M . In §3, all this formalism will be applied to the case of the geodesic spray of a Finsler metric F .

2.1. Notations. The following notations and definitions will be used throughout this work:

- Given an open set $\mathcal{O} \subseteq M$ and a smooth curve $\lambda : I \subseteq \mathbb{R} \rightarrow M$, $\mathfrak{X}(\mathcal{O})$ and $\mathfrak{X}(\lambda)$ will denote, respectively, the space of smooth vector fields on \mathcal{O} and along λ .
- $\pi : TM \setminus 0 \rightarrow M$ will denote the tangent bundle without the null section.
- The *vertical distribution* on $TM \setminus 0$ is $w \mapsto \mathcal{V}_w TM = \ker(d\pi(w))$. Vectors (or vector fields) tangent to $\mathcal{V}TM$ are called vertical.

- The *vertical tangent bundle*

$$(2.1) \quad p : \mathcal{V}TM \rightarrow TM \setminus 0$$

is obtained by restricting to $\mathcal{V}TM$ the projection map $T(TM \setminus 0) \rightarrow TM \setminus 0$.

- The *vertical lift* at $w \in TM \setminus 0$ is the tautological isomorphism

$$i_w : T_{\pi(w)}M \rightarrow \mathcal{V}_wTM, \quad i_w(u) = (d/dt)|_{t=0}(w + t \cdot u).$$

Through these maps, to any vector field U defined along a map $f : \Sigma \rightarrow M$, we can associate a vector field U^\flat defined along any lift $\bar{f} : \Sigma \rightarrow TM \setminus 0$ of f (i.e. $\pi \circ \bar{f} = f$), called the *vertical lift of U (along \bar{f})*.

- The *canonical vector field* on $TM \setminus 0$ is the vertical vector field C defined by $C(w) = i_w(w)$.
- The *almost-tangent structure* of $TM \setminus 0$ is the vector-bundle endomorphism

$$\mathcal{J} : T(TM \setminus 0) \rightarrow T(TM \setminus 0), \quad \mathcal{J}(X) = i_w(d\pi(X))$$

for $X \in T_w(TM \setminus 0)$. Note that any vertical smooth vector field on $TM \setminus 0$ (i.e. any smooth section of (2.1)) can be written, non-uniquely, as $\mathcal{J}(X)$ for some $X \in \mathfrak{X}(TM \setminus 0)$.

2.2. Sprays and connections.

Definition 2.1. A second order differential equation on M is a smooth vector field S on $TM \setminus 0$ such that $\mathcal{J}(S) = C$. This means that the integral curves of S are of the form $t \mapsto \dot{\gamma}(t)$, for some class of curves $\{\gamma\}$ in M called the geodesics of S . If furthermore $[C, S] = S$, then S is called a spray.

For future reference, we state the following straightforward lemma (see [6]).

Lemma 2.2. *If S is a second order differential equation on M and $X \in \mathfrak{X}(TM \setminus 0)$ is vertical, then $\mathcal{J}[X, S] = X$.*

Definition 2.3. A connection on M in the sense of Grifone, or simply a connection on M , is a smooth 1-form Γ on $TM \setminus 0$ with values in TM such that $-\Gamma$ is a reflexion across $\mathcal{V}TM$, i.e. $\Gamma^2 = \mathbf{I}$ and $\ker(\Gamma + \mathbf{I}) = \mathcal{V}TM$. We say that Γ is 1-homogeneous if $[C, \Gamma] = 0$.

A connection Γ on M determines an Ehresmann connection on $\pi : TM \setminus 0 \rightarrow M$ by defining $\mathcal{H}TM = \ker(\Gamma - \mathbf{I})$, so that

$$(2.2) \quad T(TM \setminus 0) = \mathcal{H}TM \oplus \mathcal{V}TM.$$

The corresponding projection operators will be denoted by \mathcal{H} and \mathcal{V} , and vectors (resp. vector fields) tangent to $\mathcal{H}TM$ will be called *horizontal*. We denote by U^\flat the *horizontal lift* of a vector field $U \in \mathfrak{X}(M)$, which is the horizontal vector field on $TM \setminus 0$ such that $d\pi(U^\flat) = U$.

By a result of Grifone [6], a spray canonically determines a connection on the manifold:¹

Theorem 2.4. *A spray S on M determines a unique symmetric and 1-homogeneous connection Γ relatively to which S is horizontal. It is called the canonical connection associated to S and is given by $\Gamma_S = -[S, \mathcal{J}]$.²*

¹We refer to [6] for the definition of a symmetric connection.

²Here, $[S, \mathcal{J}]$ stands for the Lie derivative of the tensor field \mathcal{J} along S .

Until the end of this section, we let be fixed a spray S on M with associated canonical connection Γ_S .

2.3. Linear connections on the vertical tangent bundle. Following Grifone [7], we define

Definition 2.5. Let ∇ be a linear connection on $p : \mathcal{V}TM \rightarrow TM \setminus 0$. Then,

- (1) We call ∇ *almost-projectable* if $\nabla_X C = X$ for all X vertical.
- (2) We say that ∇ *projects onto* Γ_S , or that it is a *lift* of Γ_S , if ∇ is almost-projectable and $\ker(\varphi) = \mathcal{H}TM$, where

$$\varphi : T(TM \setminus 0) \rightarrow \mathcal{V}TM, \quad \varphi(X) = \nabla_X C.$$

Thus, ∇ is a lift of Γ_S precisely when $\nabla_X C = \mathcal{V}X$, for all $X \in T(TM \setminus 0)$.

The appropriate notion of torsion for linear connections on (2.1) is given by the following (see [2]).

Definition 2.6. The *torsion* of a linear connection ∇ on $p : \mathcal{V}TM \rightarrow TM \setminus 0$ is the $\mathcal{V}TM$ -valued tensor field on $TM \setminus 0$ given by

$$T(X, Y) = \nabla_X \mathcal{J}(Y) - \nabla_Y \mathcal{J}(X) - \mathcal{J}[X, Y], \quad X, Y \in \mathfrak{X}(TM \setminus 0).$$

2.3.1. Nullity conditions for the torsion. We will be interested in the lifts ∇ of Γ_S whose torsion satisfies one of the following conditions (actually, the first one will suffice; we have included the other ones as they are fulfilled by the classical connections discussed in §4)

- (T1) $T(S, Y) = 0$ for all Y .
- (T2) $T(\mathcal{H}X, \mathcal{H}Y) = 0$ for all X, Y .
- (T3) $T(X, Y) = 0$ for all X, Y .

Next we show that, *as long as (T1) holds, applying ∇ in the direction of S does not depend on the choice of ∇ .*

Lemma 2.7. *Let ∇ be a lift of Γ_S .*

- (1) *If Y is vertical, $T(S, Y) = 0$. Therefore, (T2) \Rightarrow (T1).*
- (2) *If (T1) holds, then $\nabla_S \mathcal{J}(Y) = \mathcal{V}[S, \mathcal{J}(Y)]$ for all $Y \in \mathfrak{X}(TM \setminus 0)$.*

Proof. (1) From $\mathcal{J}(Y) = 0$ and $\mathcal{J}(S) = C$ we obtain $T(S, Y) = -\nabla_Y C - \mathcal{J}[S, Y]$, which, due to the almost-projectability of ∇ , is equal to $-Y - \mathcal{J}[S, Y]$. Now the claim follows from Lemma 2.2. (2) From $\nabla_Y \mathcal{J}(S) = \nabla_Y C = \mathcal{V}Y$, we obtain $0 = T(S, Y) = \nabla_S \mathcal{J}(Y) - \mathcal{V}Y - \mathcal{J}[S, Y]$. Now, from $\mathcal{V}Y = -\mathcal{V}\Gamma_S(Y)$ and $\Gamma_S = -[S, \mathcal{J}]$, we obtain, respectively

$$\begin{aligned} \mathcal{V}Y + \mathcal{J}[S, Y] &= \mathcal{V}(-\Gamma_S(Y) + \mathcal{J}[S, Y]) \\ &= \mathcal{V}([S, \mathcal{J}](Y) + \mathcal{J}[S, Y]) \\ &= \mathcal{V}[S, \mathcal{J}(Y)]. \end{aligned}$$

□

Until the end of this section, let be fixed a lift ∇ of Γ_S .

2.3.2. *The tensors \mathcal{C} and \mathcal{C} .* When dealing with the transition to the corresponding family of affine connections in §2.4, it will prove useful to consider the following “vertical part” of the torsion T .

Definition 2.8. (1) We define a $\mathcal{V}TM$ -valued tensor field \mathcal{C} on $TM \setminus 0$ by $\mathcal{C}(X, Y) = T(\mathcal{J}(X), Y)$.
 (2) For each $w \in TM \setminus 0$, we define $\mathcal{C}_w : T_{\pi(w)}M \times T_{\pi(w)}M \rightarrow T_{\pi(w)}M$ by $\mathcal{C}_w(u, v) = i_w^{-1}\mathcal{C}(X, Y)$, where $X, Y \in T_wTM$ are any lifts of u, v . The well-definability of \mathcal{C}_w will follow from the lemma below.

Lemma 2.9. *We have*

- (1) *The tensor field \mathcal{C} is semi-basic, i.e., $\mathcal{C}(X, Y) = 0$ whenever at least one of X, Y is vertical.*
- (2) *$\mathcal{C}(\cdot, S) = 0$. Equivalently, $\mathcal{C}_w(\cdot, w) = 0$ for all w .*

Proof. For (1), just note that

$$(2.3) \quad \nabla_{\mathcal{J}(X)} \mathcal{J}(Y) = \mathcal{J}[\mathcal{J}(X), Y] + \mathcal{C}(X, Y),$$

and that $[\mathcal{J}(X), Y]$ is vertical and, thus, $\mathcal{J}[\mathcal{J}(X), Y] = 0$, whenever Y is vertical. (2) is just a reformulation of (1) of Lemma 2.7 since T is skew-symmetric. \square

2.3.3. *The curvature endomorphisms of a spray.* Denote by \mathcal{R} the curvature tensor of ∇ ,

$$\mathcal{R}(X, Y)\mathcal{J}(Z) = \nabla_{[X, Y]}\mathcal{J}(Z) - [\nabla_X, \nabla_Y]\mathcal{J}(Z).$$

Our aim here is to show that the following definition is intrinsic to the spray S .

Definition 2.10. Given $w \in TM \setminus 0$, the *curvature endomorphism in the direction w* is the map $\mathbf{R}_w : T_{\pi(w)}M \rightarrow T_{\pi(w)}M$,

$$\mathbf{R}_w(u) = i_w^{-1}\mathcal{R}(w^h, u^h)w^v = i_w^{-1}\mathcal{R}(S, u^h)C$$

Lemma 2.11. *We have*

- (1) *If $X \in \mathfrak{X}(TM \setminus 0)$ is horizontal, then $\mathcal{R}(S, X)C = \mathcal{V}[S, X]$.*
- (2) *If (T1) holds and if X is vertical, then $\mathcal{R}(S, X)C = 0$.*

Proof. (1) Let $X \in \mathfrak{X}(TM \setminus 0)$ be horizontal. As S is horizontal as well and ∇ is a lift of Γ_S , then $\nabla_S C = \nabla_X C = 0$ and $\nabla_{[S, X]}C = \mathcal{V}[S, X]$. Therefore, $\mathcal{R}(S, X)C = \mathcal{V}[S, X]$. (2) If X is vertical, then $\nabla_X C = X$ and thus $\mathcal{R}(S, X)C = \mathcal{V}[S, X] - \nabla_S X$. If, furthermore, (T1) holds, from Lemma 2.7 we obtain $\nabla_S X = \mathcal{V}[S, X]$ and therefore $\mathcal{R}(S, X)C = 0$. \square

Corollary 2.12. *The curvature endomorphisms \mathbf{R}_w , for $w \in TM \setminus 0$, do not depend on the choice of the lift ∇ , but only on the spray S . Furthermore, if (T1) holds then for any lift X of u at w ,*

$$(2.4) \quad \mathbf{R}_w(u) = i_w^{-1}\mathcal{R}(S, X)C.$$

2.4. The associated family of affine connections on M .

Definition 2.13. Let $w \in TM \setminus 0$. If $\mathcal{O} \subseteq M$ is an open neighborhood of $x = \pi(w)$, we define

$$D^w \cdot : T_x M \times \mathfrak{X}(\mathcal{O}) \rightarrow T_x M, \quad D_u^w V = i_w^{-1}\nabla_{u^h} V^v,$$

where u^h is the horizontal lift of u at w .

The map $D^w \cdot : T_x M \times \mathfrak{X}(\mathcal{O}) \rightarrow T_x M$ just defined is \mathbb{R} -linear in u and V and satisfies the Leibniz rule for V . In particular, for each smooth curve $\lambda : I \rightarrow M$ through x at $t = t_0$ it induces a map $D^w/dt : \mathfrak{X}(\lambda) \rightarrow T_x M$ satisfying the properties of a covariant derivative. In terms of ∇ ,

$$(2.5) \quad \frac{D^w V}{dt} = i_w^{-1} \left(\frac{\nabla V^v}{dt} \right) (t_0),$$

where $V^v(t)$ is the vertical lift of $V(t)$ along the horizontal lift $\bar{\lambda} : I \rightarrow TM \setminus 0$ of $\lambda : I \rightarrow M$ through w at $t = t_0$.

By considering $W \in \mathfrak{X}(\mathcal{O})$ (resp., $W(t) \in \mathfrak{X}(\lambda)$, for some smooth curve λ) nowhere null, we thus obtain well-defined maps

$$\begin{aligned} D^W \cdot : \mathfrak{X}(\mathcal{O}) \times \mathfrak{X}(\mathcal{O}) &\rightarrow \mathfrak{X}(\mathcal{O}) \\ D^W/dt : \mathfrak{X}(\lambda) &\rightarrow \mathfrak{X}(\lambda) \end{aligned}$$

satisfying the properties of being an affine connection on \mathcal{O} and a covariant derivative along λ , respectively.

Definition 2.14. We call $\{D^W\}_{(\mathcal{O}, W)}$ the family of affine connections corresponding to ∇ .

Proposition 2.15. Let $W \in \mathfrak{X}(\mathcal{O})$, nowhere null, and $\lambda : I \rightarrow M$ a regular curve on M .

- (1) If **(T1)** holds, then the map $D^W \cdot : \mathfrak{X}(\mathcal{O}) \rightarrow \mathfrak{X}(\mathcal{O})$ does not depend on the choice of ∇ , but only on the spray S ; equivalently, if **(T1)** holds, the map $D^\lambda/dt : \mathfrak{X}(\lambda) \rightarrow \mathfrak{X}(\lambda)$ is intrinsic to the spray S .
- (2) If **(T3)** holds, then we could have chosen any lift of u at w in Definition 2.13 (resp., any lift of λ through w at $t = t_0$ in (2.5)).

Proof. (1) Along $W(\mathcal{O}) \subset TM \setminus 0$, W^h coincides with S . So, $D_W^W U = i_W^{-1} \nabla_S U^v$ which according to (2) of Lemma 2.7, does not depend on the choice of ∇ . For (2), we have to show that $\nabla_X V^v = 0$ if X is vertical and **(T3)** holds. Choosing $X \in \mathfrak{X}(TM \setminus 0)$ vertical in $T(X, Y) = 0$ we obtain $\nabla_X \mathcal{J}(Y) = \mathcal{J}[X, Y]$. If, furthermore, Y is projectable, then $[X, Y]$ is vertical and hence $\mathcal{J}[X, Y] = 0$. So, for X vertical and Y projectable, $\nabla_X \mathcal{J}(Y) = 0$. Now, just note that $U^v = \mathcal{J}(Y)$ for any lift Y of U . \square

We leave to §2.4.2 the proof of the following.

Lemma 2.16. A regular curve $\lambda : I \rightarrow M$ is a geodesic of S if, and only if, $\frac{D^\lambda \dot{\lambda}}{dt} = 0$ for all t .

2.4.1. Symmetry.

Proposition 2.17. Let $W \in \mathfrak{X}(\mathcal{O})$ be nowhere null.

- (1) If **(T1)** holds, then $D_W^W V - D_V^W W = [W, V]$ for all $V \in \mathfrak{X}(\mathcal{O})$.
- (2) If **(T2)** holds, then $D_U^W V - D_V^W U = [U, V]$ for all $U, V \in \mathfrak{X}(\mathcal{O})$.

Proof. As $U^v = \mathcal{J}(U^h)$ and $V^v = \mathcal{J}(V^h)$, then $D_U^W V - D_V^W U = i_W^{-1} (\nabla_{U^h} \mathcal{J}(V^h) - \nabla_{V^h} \mathcal{J}(U^h)) = i_W^{-1} T(U^h, V^h) + i_W^{-1} \mathcal{J}[U^h, V^h]$. But $d\pi[U^h, V^h] = [U, V]$, hence $i_W^{-1} \mathcal{J}[U^h, V^h] = [U, V]$. To conclude, just note that, along $W(\mathcal{O}) \subset TM \setminus 0$, $W^h = S$. \square

As usual, the above symmetry properties of D^W implies the following symmetry between covariant derivatives

$$(2.6) \quad \frac{D^T T}{ds} = \frac{D^T U}{dt}$$

whenever **(T1)** holds; here, $T = \partial H / \partial t$ and $U = \partial H / \partial s$ are the coordinate vector fields along a smooth variation $H : (-\varepsilon, \varepsilon) \times I \rightarrow M$ of regular curves (i.e. each curve $t \mapsto H(s, t)$ is regular), and D^T/dt and D^T/ds are, respectively, the covariant derivatives along the curves $t \mapsto H(s, t)$ and $s \mapsto H(s, t)$, corresponding to T .

2.4.2. Working with non-horizontal lifts. Given $W, U \in \mathfrak{X}(\mathcal{O})$ (resp., $W, U \in \mathfrak{X}(\lambda)$), with W nowhere null, $dW(U) : \mathcal{O} \rightarrow T(TM \setminus 0)$ (resp., $W : I \rightarrow TM \setminus 0$) is a natural lift of U (resp., of λ) defined along, and tangent to, the submanifold $W(\mathcal{O}) \subset TM \setminus 0$. It will be desirable to use these lifts, instead of the horizontal ones, to compute $D_U^W V$ and $D^W V/dt$.

Lemma 2.18. *Within the above notation, we have (recall §2.3.2)*

$$(2.7) \quad D_U^W V = i_W^{-1} \nabla_{dW(U)} V^v - \mathcal{C}_W(D_U^W W, V).$$

$$(2.8) \quad \frac{D^W V}{dt} = i_W^{-1} \frac{\nabla V^v}{dt} - \mathcal{C}_W\left(\frac{D^W W}{dt}, V\right)$$

In particular, as $\mathcal{C}_W(\cdot, W) = 0$, then $D_U^W W = i_W^{-1} \nabla_{dW(U)} W^v$. (In (2.8), V^v stands for the vertical lift of $V \in \mathfrak{X}(\lambda)$ along the curve $W : I \rightarrow TM \setminus 0$.)

Proof. We will only consider (2.7), (2.8) being analog. Since $[\mathcal{J}(X), Y]$ is vertical if Y is projectable, we obtain from (2.3) that $\nabla_{\mathcal{J}(X)} \mathcal{J}(Y) = \mathcal{C}(X, Y)$ whenever Y is projectable. It follows that $\nabla_{\mathcal{V}X} V^v = \mathcal{C}(\hat{X}, V^h)$ for all \hat{X} such that $\mathcal{J}(\hat{X}) = \mathcal{V}X$. Applying this to $X = dW(U)$, and noting that $\mathcal{H}dW(U) = U^h$, we obtain $\nabla_{dW(U)} V^v = \nabla_{\mathcal{H}dW(U)} V^v + \nabla_{\mathcal{V}dW(U)} V^v = i_W D_U^W V + \mathcal{C}(\hat{X}, V^h)$. Now, from $\mathcal{J}(\hat{X}) = \mathcal{V}dW(U)$ we obtain $d\pi(\hat{X}) = i_W^{-1} \mathcal{V}dW(U)$ and, therefore, $\mathcal{C}(\hat{X}, V^h) = i_W \mathcal{C}_W(i_W^{-1} \mathcal{V}dW(U), V)$. It remains to show that

$$(2.9) \quad i_W^{-1} \mathcal{V}dW(U) = D_U^W W.$$

For such, let $V = W$ in $\nabla_{dW(U)} V^v = i_W D_U^W V + \mathcal{C}(\hat{X}, V^h)$ and observe that $\nabla_{dW(U)} W^v = \mathcal{V}dW(U)$, since $W^v|_{W(\mathcal{O})} = C|_{W(\mathcal{O})}$, and $\mathcal{C}(\hat{X}, W^h) = 0$ since $W^h|_{W(\mathcal{O})} = S|_{W(\mathcal{O})}$. \square

Remark 2.19. For future reference, we remark that, analogously to (2.9), we have $D^W W/dt = i_W^{-1} \mathcal{V}dW/dt$, for any nowhere null $W \in \mathfrak{X}(\lambda)$.

Proof of Lemma 2.16. The vertical lift $\dot{\lambda}^v$ of $\dot{\lambda}$ along the curve $t \mapsto \dot{\lambda}(t)$ coincides with C , so substituting $W(t) = V(t) = \dot{\lambda}(t)$ in (2.8), and recalling that $\mathcal{C}_{\dot{\lambda}}(\cdot, \dot{\lambda}) = 0$, we obtain $D^{\dot{\lambda}} \dot{\lambda}/dt = i_{\dot{\lambda}}^{-1} \nabla \dot{\lambda}^v/dt = i_{\dot{\lambda}}^{-1} \nabla C/dt$ which is null if, and only if, the curve $t \mapsto \dot{\lambda}(t)$ is horizontal, which in turn is equivalent to it being an integral line of S . \square

2.4.3. *The curvature endomorphisms of the affine connections.* We can now prove the main result of §2, which shows how one can compute the curvature endomorphisms, defined in §2.3.3, by means of the covariant derivatives.

Theorem 2.20. *Suppose that (T1) holds. Let $H : (-\varepsilon, \varepsilon) \times I \rightarrow M$ be a smooth variation of a curve $\lambda(t) = H(0, t)$ through regular curves (i.e., each $t \mapsto H(s, t)$ is regular). Following the same notation as §2.4.1, if λ is a geodesic then, at any $(0, t)$,*

$$\frac{D^T}{ds} \frac{D^T T}{dt} - \frac{D^T}{dt} \frac{D^T T}{ds} = \mathbf{R}_T(U).$$

In particular, if the variation is through geodesics, the variational vector field $J(t) = U(0, t)$ will satisfy the Jacobi equation

$$\frac{D^{\dot{\lambda}} J}{dt^2} + \mathbf{R}_{\dot{\lambda}}(J) = 0$$

Instead of proving this theorem, we will prove the following somewhat more abstract version from which Theorem 2.20 follows easily.

Proposition 2.21. *Suppose that (T1) holds. Let $W \in \mathfrak{X}(\mathcal{O})$, nowhere null, be such that its integral curve through some point x_0 is a geodesic. Then, at x_0 , the curvature endomorphism in the direction W of the affine connection D^W coincides with \mathbf{R}_W ; that is, given $U \in \mathfrak{X}(\mathcal{O})$, then, at x_0 ,*

$$D_U^W D_W^W W - D_W^W D_U^W W + D_{[W, U]}^W W = \mathbf{R}_W(U)$$

Proof. To simplify the notation, let $\widetilde{W} = dW(W)$ and $\widetilde{U} = dW(U)$ be the vector fields tangent to $W(\mathcal{O}) \subset TM \setminus 0$. According to Lemma 2.18, along $W(\mathcal{O})$ we have $(D_W^W W)^\flat = \nabla_{\widetilde{W}} W^\flat$. Hence, applying Lemma 2.18 again,

$$(2.10) \quad D_U^W D_W^W W = i_W^{-1} \nabla_{\widetilde{U}} \nabla_{\widetilde{W}} W^\flat - \mathcal{C}_W(D_U^W W, D_W^W W).$$

Arguing similarly,

$$(2.11) \quad D_W^W D_U^W W = i_W^{-1} \nabla_{\widetilde{W}} \nabla_{\widetilde{U}} W^\flat - \mathcal{C}_W(D_W^W W, D_U^W W).$$

Also, from $dW([W, U]) = [\widetilde{W}, \widetilde{U}]$ and Lemma 2.18, we have $D_{[W, U]}^W W = i_W^{-1} \nabla_{[\widetilde{W}, \widetilde{U}]} W^\flat$. Together with (2.10) and (2.11), this gives us

$$\begin{aligned} D_U^W D_W^W W - D_W^W D_U^W W + D_{[W, U]}^W W &= i_W^{-1} \mathcal{R}(\widetilde{W}, \widetilde{U}) W^\flat \\ &\quad + \mathcal{C}_W(D_W^W W, D_U^W W) - \mathcal{C}_W(D_U^W W, D_W^W W). \end{aligned}$$

Now, $W^\flat|_{W(\mathcal{O})} = C$, and the hypothesis on W implies that, at $W(x_0)$, $\widetilde{W} = S$. Therefore, as (T1) holds, and $d\pi(\widetilde{U}) = U$, relation (2.4) guaranties that, at x_0 , $i_W^{-1} \mathcal{R}(\widetilde{W}, \widetilde{U}) W^\flat = \mathbf{R}_W(U)$. To conclude, just note that the hypothesis on W implies $(D_W^W W)(x_0) = 0$. \square

3. THE CASE OF A FINSLER METRIC

We now turn back to the case where we are given a Finsler metric F on M . As it is well-known (see [1]), the solutions to the Euler-Lagrange equations of (1.1) are the geodesics of a second order differential equation S on M defined via

$$dE = \omega_F(S, \cdot),$$

where ω_F is the symplectic structure on $TM \setminus 0$ induced by F (see the Appendix). The homogeneity of F guarantees that S is in fact a spray, the so called *geodesic spray* of F .

Throughout this section, let be fixed a lift ∇ of Γ_S satisfying condition **(T1)**.

3.1. Metric conditions on ∇ . By abuse of notation we still denote by g the Riemannian metric on (2.1) induced by the fundamental tensor of F with the help of the vertical lift. We start by observing that g is always parallel in the direction of S :

Lemma 3.1. *We have $\nabla_S g = 0$.*

Proof. We already know from Lemma 2.7 that this covariant derivative does not depend on the choice of ∇ . In particular, we can assume that ∇ is any one of the classical connections discussed in §4, in which case the result follows immediately from the properties of the connection. \square

Definition 3.2. We denote by \mathcal{C}_b the following metric contraction of \mathcal{C} ,

$$\mathcal{C}_b(X, Y, Z) = g(\mathcal{C}(X, Y), \mathcal{J}(Z)).$$

Also, for each $w \in TM \setminus 0$, we define $(\mathcal{C}_b)_w : T_{\pi(w)}M \times T_{\pi(w)}M \times T_{\pi(w)}M \rightarrow \mathbb{R}$ by $(\mathcal{C}_b)_w(u, v, t) = i_w^{-1} \mathcal{C}_b(X, Y, Z) = g_w(\mathcal{C}_w(u, v), t)$, where $X, Y, Z \in T_w TM$ are any lifts of u, v, t , respectively.

The metric conditions on ∇ we will be interested in can now be stated as

$$\textbf{(M1)} \quad (\nabla \cdot g)(\cdot, C) = 0.$$

$$\textbf{(M2)} \quad \mathcal{C}_b(\cdot, \cdot, S) = 0.$$

Note that condition **(M2)** may be written as $(\mathcal{C}_b)_w(\cdot, \cdot, w) = 0$ for all $w \in TM \setminus 0$.

3.2. Metric properties of the affine connections. Back to the family of affine connections D^W , we can state

Proposition 3.3. *Let $W, U, V, T \in \mathfrak{X}(\mathcal{O})$, with W nowhere null, where $\mathcal{O} \subseteq M$ is an open set.*

(i) *If **(M1)** and **(M2)** hold, then*

$$Ug_W(W, V) = g_W(D_U^W W, V) + g_W(W, D_U^W V).$$

(ii) *If the integral curve of W through some point x_0 is a geodesic, then, at x_0 ,*

$$Wg_W(T, V) = g_W(D_W^W T, V) + g_W(T, D_W^W V).$$

Proof. Computing $\nabla_{dW(U)} T^\flat$ and $\nabla_{dW(U)} V^\flat$ with the help of Lemma 2.18,

$$\begin{aligned} (3.1) \quad (\nabla_{dW(U)} g)(T^\flat, V^\flat)|_{W(x)} &= (D_U^W g_W)(T, V)|_x - (\mathcal{C}_b)_W(D_U^W W, T, V)|_x \\ &\quad - (\mathcal{C}_b)_W(D_U^W W, V, T)|_x \end{aligned}$$

(i) Let $T = W$ in (3.1). As $W^\flat|_{W(\mathcal{O})} = C|_{W(\mathcal{O})}$, the left-hand side of (3.1) vanishes since **(M1)** holds. Also, (2) of Lemma 2.9 gives $(\mathcal{C}_b)_W(D_U^W W, T, V) = 0$, whereas the hypothesis **(M2)** gives $(\mathcal{C}_b)_W(D_U^W W, V, T) = 0$.

(ii) Let $U = W$ in (3.1). The hypothesis on W implies $dW(W)|_{W(x_0)} = S|_{W(x_0)}$ and hence, from Lemma 3.1, $(\nabla_{dW(U)} g)(T^\flat, V^\flat)|_{W(x_0)} = 0$. Also, $(D_W^W W)(x_0) = 0$ and the result follows. \square

- Remark 3.4.* (1) Observe that although we have assumed the validity of **(T1)** throughout this section, property (i) of Proposition 3.3 does not require that hypothesis.
- (2) As usual, Proposition 3.3 implies analogous statements for covariant derivatives which, for the sake of brevity, we will not enunciate here.

3.3. The second variation of energy and the second fundamental form.
Throughout this subsection, we assume that conditions (M1) and (M2) hold.

Let be fixed two submanifolds $\mathcal{P}_1, \mathcal{P}_2 \subset M$ and denote by $\Omega_{\mathcal{P}_1, \mathcal{P}_2}$ the set of all regular curves $\lambda : I = [0, 1] \rightarrow M$ joining \mathcal{P}_1 to \mathcal{P}_2 . The corresponding energy functional is

$$(3.2) \quad E : \Omega_{\mathcal{P}_1, \mathcal{P}_2} \longrightarrow \mathbb{R}, \quad E(\lambda) = \frac{1}{2} \int_0^1 F(\dot{\lambda}(t))^2 dt.$$

When computing the second variation of (3.2) at a critical point, a type of second fundamental form will come into play. Firstly, for a submanifold $\mathcal{P} \subset M$ we define

Definition 3.5. Given $x \in \mathcal{P}$, we say that a vector $\eta \in T_x M \setminus 0$ is *normal* to \mathcal{P} if $g_\eta(\eta, u) = 0$ for all $u \in T_x \mathcal{P}$. The set of all such vectors forms a cone in $T_x M \setminus 0$, called *the normal cone to \mathcal{P} at x* , and denoted by $\nu_x(\mathcal{P})$.

Definition 3.6. Given $\eta \in \nu_x(\mathcal{P})$, the *second fundamental form of \mathcal{P} relative to the normal direction η* is the map

$$h_\eta^\mathcal{P} : T_x \mathcal{P} \times T_x \mathcal{P} \rightarrow \mathbb{R}, \quad h_\eta^\mathcal{P}(u, v) = \frac{1}{2} g_\eta(\eta, D_U^\eta V + D_V^\eta U),$$

where $U, V \in \mathfrak{X}(\mathcal{P})$ are any extensions of u, v . Of course, this is a well-defined symmetric bilinear form.

- Remark 3.7.* (1) We remark that, as a consequence of Proposition 2.15 and the theorem below, $h_\eta^\mathcal{P}$ does not depend on the choice of ∇ as long as conditions **(T1)**, **(M1)** and **(M2)** hold. Indeed, this is also the case even if we drop assumption **(T1)** as we show in Proposition 5.3.
- (2) If, in addition, **(T2)** holds, then it follows from Proposition 2.17 that $h_\eta^\mathcal{P}(u, v) = g_\eta(\eta, D_U^\eta V)$.

Theorem 3.8. *Let $s \mapsto \lambda_s \in \Omega_{\mathcal{P}_1, \mathcal{P}_2}$ be a smooth 1-parameter family starting at a geodesic $\lambda = \lambda_0$ normal to \mathcal{P}_1 and \mathcal{P}_2 at its extremes. Denoting by V the variational vector field along λ , we have*

$$(3.3) \quad \begin{aligned} \frac{d^2}{ds^2} \Big|_{s=0} E(\lambda_s) &= \int_0^1 g_\lambda \left(\frac{D^\lambda V}{dt}, \frac{D^\lambda V}{dt} \right) - g_\lambda(\mathbf{R}_\lambda(V), V) dt \\ &\quad + h_{\lambda(1)}^{\mathcal{P}_2}(V(1), V(1)) - h_{\lambda(0)}^{\mathcal{P}_1}(V(0), V(0)). \end{aligned}$$

Proof. Once we have at hand Proposition 3.3, Theorem 2.20 and the symmetry relation (2.6), the proof proceeds exactly as in the Riemannian case since the homogeneity of F implies the relation $F(w)^2 = g_w(w, w)$. We only remark that at some point it will be necessary to assure that

$$\frac{\partial}{\partial s} g_T \left(\frac{D^T T}{dt}, U \right) \Big|_{s=0} = g_T \left(\frac{D^T}{ds} \frac{D^T T}{dt}, U \right) \Big|_{s=0}.$$

(following the notation in Theorem 2.20). Now, this equality is true regardless of any assumptions on ∇ as follows from the fact that $(D^T T/dt)(0, t) = 0$ for all t and the following trivial fact that just reflects the tensorial nature of $D^W g_W$:

Given $W, U, V, T \in \mathfrak{X}(\mathcal{O})$, if $V(x_0) = 0$ then, at x_0 , $Tg_W(V, U) = g_W(D_T^W V, U)$ \square

4. THE CONNECTIONS OF BERWALD, CARTAN, CHERN-RUND AND HASHIGUCHI

4.1. The Berwald connection of a spray. The Berwald connection is defined for any spray S regardless to whether or not S is the geodesic spray of a Finsler metric. We take from [7] the following expressions.

Proposition-Definition 4.1. *The following*

$$\begin{aligned}\nabla_{\mathcal{J}(X)}^* \mathcal{J}(Y) &= \mathcal{J}[\mathcal{J}(X), Y] \\ \nabla_{\mathcal{H}X}^* \mathcal{J}(Y) &= \mathcal{V}[\mathcal{H}X, \mathcal{J}(Y)]\end{aligned}$$

defines a lift ∇^* of Γ_S , called the Berwald connection of S .

Let \mathcal{F} denote the almost-complex structure on $TM \setminus 0$ characterized by $\mathcal{F}\mathcal{J} = \mathcal{H}$ and $\mathcal{F}\mathcal{H} = -\mathcal{J}$. Having in mind (2.3), a direct computation shows that any linear connection ∇ on (2.1) may be written as

$$\nabla_X \mathcal{J}(Y) = \nabla_X^* \mathcal{J}(Y) + \mathcal{C}(\mathcal{F}X, Y) + \mathcal{C}'(X, Y),$$

where $\mathcal{C}'(X, Y) = \nabla_{\mathcal{H}X} \mathcal{J}(Y) - \mathcal{V}[\mathcal{H}X, \mathcal{J}(Y)]$ defines a semi-basic $\mathcal{V}TM$ -valued tensor field on $TM \setminus 0$. Actually, one can show the following

Proposition 4.2. *There is a 1-1 correspondence between the lifts of Γ_S and the pairs of $\mathcal{V}TM$ -valued semi-basic tensor fields \mathcal{C} and \mathcal{C}' on $TM \setminus 0$ satisfying $\mathcal{C}(\cdot, S) = \mathcal{C}'(\cdot, S) = 0$.*

4.2. The connections of Berwald (bis), Cartan, Chern-Rund and Hashiguchi.

Now, let us suppose that S is the geodesic spray of a Finsler metric F .

Definition 4.3. For each $w \in TM \setminus 0$, the *Cartan tensor* of F at w is the 3-linear map $C_w : T_{\pi(w)}M \times T_{\pi(w)}M \times T_{\pi(w)}M \rightarrow \mathbb{R}$,

$$C_w(u, v, z) = \frac{1}{4} \frac{\partial^3}{\partial r \partial s \partial t} \Big|_{r=s=t=0} F(w + ru + sv + tz)^2.$$

Next we define the tensor C' . For that, let D^W be the affine connections associated to any lift ∇ of Γ_S satisfying **(T1)** (e.g. the Berwald lift).

Definition 4.4. Given $w \in TM \setminus 0$, we define a 3-linear map $C'_w : T_{\pi(w)}M \times T_{\pi(w)}M \times T_{\pi(w)}M \rightarrow \mathbb{R}$ as

$$C'_w(u, v, z) = (D_W^W C_W)|_{\pi(w)}(u, v, z),$$

where W is any nowhere null vector field defined around $\pi(w)$ whose integral curve through $\pi(w)$ is a geodesic.

Lemma 4.5. *For each $w \in TM \setminus 0$, C_w and C'_w are fully symmetric and $C_w(\cdot, \cdot, w) = C'_w(\cdot, \cdot, w) = 0$.*

Proof. The statements about C are well-known (e.g. [3]), whereas the ones about C' follow easily from those. \square

Let us consider the following conditions for a linear connection ∇ on (2.1)

- (M3) $(\nabla_X g)(\cdot, \cdot) = 2C(\mathcal{V}X, \cdot, \cdot)$
- (M4) $(\nabla_X g)(\cdot, \cdot) = 2C'(\mathcal{J}(X), \cdot, \cdot)$
- (M5) $(\nabla_X g)(\cdot, \cdot) = 2C(\mathcal{V}X, \cdot, \cdot) + 2C'(\mathcal{J}(X), \cdot, \cdot)$
- (M6) $\nabla g = 0$
- (M7) $\mathcal{C}_b(X, Y, Z) = \mathcal{C}_b(X, Z, Y)$, for all X, Y, Z ,

where we have tacitly identified C and C' with tensor fields on (2.1) via the vertical lift. It follows from Lemma 4.5 that each one of (M3), (M4), (M5) implies (M1), whereas (2) of Lemma 2.9 guaranties that (M7) implies (M2).

The connections of Berwald, Cartan, Chern-Rund and Hashiguchi can be described by the following existence-uniqueness results. We remark that, except for the statement concerning the Cartan connection (which is stated without proof in [2]), we were not able to find in the literature an appropriate reference for them, although one can show without difficult their equivalence with the statements in [4].

Theorem 4.6. *For each set of conditions below, there exists one, and only one, lift ∇ of Γ_S which fulfils them:*

- (1) (T3) and (M5).
- (2) (T2), (M6) and (M7).
- (3) (T3) and (M3).
- (4) (T2), (M4) and (M7).

They are called, respectively, the Berwald, Cartan, Chern-Rund and Hashiguchi connections of the Finsler metric F .

Remark 4.7. Recall from §4.1 that any ∇ is determined by the tensors \mathcal{C} and \mathcal{C}' . In terms of these, or rather in terms of \mathcal{C}_b and \mathcal{C}'_b (\mathcal{C}'_b is defined analogously to \mathcal{C}_b), the above connections are given by (compare with [5])

- (1) *Cartan:* $(\mathcal{C}_b)_w(u, v, t) = C_w(u, v, t)$ and $(\mathcal{C}'_b)_w(u, v, t) = C'_w(u, v, t)$;
- (2) *Chern-Rund:* $\mathcal{C}_b = 0$ and $(\mathcal{C}'_b)_w(u, v, t) = C'_w(u, v, t)$;
- (3) *Hashiguchi:* $(\mathcal{C}_b)_w(u, v, t) = C_w(u, v, t)$ and $\mathcal{C}'_b = 0$.

As for the families of affine connections on M corresponding to the above classical connections, we have the following theorem which was already known in the cases of Berwald and Chern-Rund connections (e.g. [12]).

Theorem 4.8. *The Cartan and Chern-Rund families of affine connections satisfy both*

$$(4.1) \quad D_U^W V - D_V^W U = [U, V]$$

$$(4.2) \quad (D_U^W g_W)(T, V) = 2C_W(D_U^W W, T, V),$$

whereas the ones corresponding to Berwald and Hashiguchi satisfy both (4.1) and

$$(4.3) \quad (D_U^W g_W)(T, V) = 2C_W(D_U^W U, T, V) + 2C'_W(U, T, V).$$

Proof. The symmetry (4.1) was already proved in Proposition 2.17 since all the connections satisfy (T2). Now, on one hand, formula (3.1) express $D^W g_W$ in terms of ∇g and \mathcal{C}_b . On the the other hand, Theorem 4.6 provides ∇g , whereas Remark 4.7 gives us the corresponding tensors \mathcal{C}_b . These observations, together with (2.9), immediatly lead to the desired relations. \square

Proposition 4.9. *Each set of properties stated in Theorem 4.8 uniquely determine a family of affine connections on M . Therefore, the Cartan and Chern-Rund (resp., Berwald and Hashiguchi) families of affine connections coincide.*

Proof. Both uniqueness statements are proved in [12]. For an alternative proof regarding the first set of conditions, see [9]. \square

5. APPENDIX: THE SECOND FUNDAMENTAL FORM FROM A SYMPLECTIC POINT OF VIEW

The notion of second fundamental form we considered in §3.3 can be regarded from a purely symplectic point of view. First of all, the Legendre transformation of F , $\mathcal{L} : TM \setminus 0 \rightarrow T^*M \setminus 0$, $\mathcal{L}(w) \cdot u = g_w(w, u)$, pulls the canonical symplectic form of T^*M back to a symplectic form ω_F on $TM \setminus 0$ which can be described via the isomorphism

$$(5.1) \quad T_w TM \simeq T_{\pi(w)}M \oplus T_{\pi(w)}M, \quad X \mapsto (d\pi(X), i_w^{-1} \mathcal{V}X)$$

as $\omega_F((u_1, v_1), (u_2, v_2)) = g_w(u_1, v_2) - g_w(v_1, u_2)$.

Given a submanifold $\mathcal{P} \subset M$, let $\nu(\mathcal{P}) \subset TM \setminus 0$ denote its *normal bundle*, i.e. the collection of all normal cones $\nu_x(\mathcal{P})$, $x \in \mathcal{P}$.

Lemma 5.1. *$\nu(\mathcal{P})$ is a Lagrangean submanifold of $(TM \setminus 0, \omega_F)$.*

Proof. Observe that, via the Legendre transformation \mathcal{L} , $\nu(\mathcal{P})$ corresponds to the conormal bundle $\text{co}(\mathcal{P}) = \{\xi \in T^*M : \pi(\xi) \in \mathcal{P} \text{ and } \xi(T_{\pi(\xi)}\mathcal{P}) = \{0\}\}$ with the zero section excluded. Now, $\text{co}(\mathcal{P})$ is a Lagrangean submanifold of T^*M since $\dim \text{co}(\mathcal{P}) = \dim M$ and the canonical 1-form of T^*M pulls back to the null form on $\text{co}(\mathcal{P})$ as can be easily verified. \square

We refer to [10, Exercise 1.17] for the following result from symplectic linear algebra.

Proposition 5.2. *Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional real vector space endowed with a non-degenerate inner product. On $V \oplus V$, consider the symplectic bilinear form $\omega((u_1, v_1), (u_2, v_2)) = \langle u_1, v_2 \rangle - \langle v_1, u_2 \rangle$. Then, there is a bijection between the set of all pairs (S, b) , with $S \subseteq V$ a linear subspace and b a symmetric bilinear form on S , and the set of all Lagrangean subspaces $L \subset (V \oplus V, \omega)$, which maps (S, b) to $L = \{(u, v) : u \in S \text{ and } \langle v, \cdot \rangle|_S + b(u, \cdot) = 0\}$.*

Observe that, in the above proposition, we must have $S = \text{pr}_1(L)$, where pr_1 is the projection onto the first summand V . Hence, the above proposition applied to the Lagrangean subspace $T_\eta \nu(\mathcal{P}) \subset (T_x M \oplus T_x M, \omega_F)$, $x = \pi(\eta)$, gives us a symmetric bilinear form b_η on $d\pi(T_\eta \nu(\mathcal{P})) = T_x \mathcal{P}$ such that

$$(5.2) \quad T_\eta \nu(\mathcal{P}) = \{(u, v) : u \in T_x \mathcal{P} \text{ and } g_\eta(v, \cdot)|_{T_x \mathcal{P}} + b_\eta(u, \cdot) = 0\}.$$

Proposition 5.3. *The form b_η coincides with the second fundamental form $h_\eta^\mathcal{P}$ constructed with the help of any lift ∇ of Γ_S satisfying (M1) and (M2).*

Proof. Given $u \in T_x \mathcal{P}$, we will show that $b_\eta(u, u) = h_\eta^\mathcal{P}(u, u)$. Let $v \in T_x M$ be such that $X := (u, v) \in T_\eta \nu(\mathcal{P})$, so that, by (5.2), $b_\eta(u, u) = g_\eta(v, u)$. Let $\theta : (-\varepsilon, \varepsilon) \rightarrow \nu(\mathcal{P})$ be any smooth curve with $\theta(0) = \eta$ and $\dot{\theta}(0) = X$. Define $H : (-\varepsilon, \varepsilon) \times (-\delta, \delta) \rightarrow M$ by $H(s, t) = \gamma_{\theta(s)}(t)$, where $\gamma_{\theta(s)} : (-\delta, \delta) \rightarrow M$

is the geodesic with $\dot{\gamma}_{\theta(s)}(0) = \theta(s)$. As $s \mapsto H(s, 0)$ is a curve on \mathcal{P} with velocity u at $t = 0$, and $T(0, 0) = \eta$, it follows from the definition of $h_\eta^\mathcal{P}$ that $h_\eta^\mathcal{P}(u, u) = g_T(D^T U/ds, T)|_{(0,0)}$. Applying (i) of Proposition 3.3 and recalling that, by construction, $g_T(U, T)|_{(s,0)} = g_{\theta(s)}(U(s, 0), \theta(s)) = 0$ for all s , we obtain $h_\eta^\mathcal{P}(u, u) = -g_T(U, D^T T/ds)|_{(0,0)} = g_\eta(u, D^\theta \theta/ds|_{s=0})$. Now, from Remark 2.19, $D^\theta \theta/ds|_{s=0} = i_{\theta(0)}^{-1} \mathcal{V} \dot{\theta}(0) = v$. Therefore, $h_\eta^\mathcal{P}(u, u) = g_\eta(u, v)$. \square

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